Timing is Money: The Impact of Arrival Order in Beta-Bernoulli Prediction Markets

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This author did the bulk of his work when he was at the University of Michigan

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Introduction

- Prediction markets are incentive-compatible mechanisms designed to **elicit the personal beliefs** of traders about a future uncertain event and **aggregate those beliefs** into the market price
- Prediction markets have been empirically observed to outperform polls as they have built-in **financial incentives** and **timely responses**
- We study the impact of traders' **informativeness**, **budget**, and the **sequence** in which they trade on **aggregation** properties and trader **compensation** under a new prediction market design

Background

- The **Bernoulli probability distribution** models an uncertain binary event with success parameter p

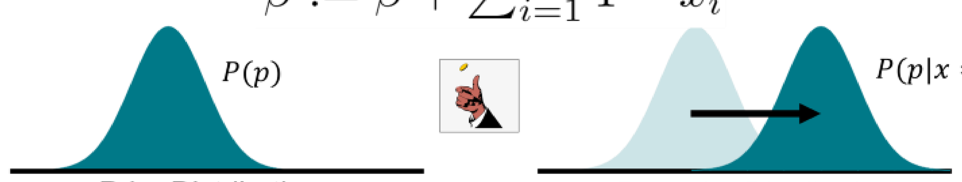
$$f(x;p) = p^x(1-p)^{1-x}, \quad \text{for } x \in 0,1$$

- The **Beta** distribution is **conjugate prior** for the Bernoulli distribution success parameter and has pdf given by

$$f(p;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}p^{\alpha-1}(1-p)^{\beta-1}$$

- Given a Beta prior over p and a sample of Bernoulli observations, the **Bayesian posterior update** is given by

$$\hat{\alpha} := \alpha + \sum_{i=1}^{N} x_i$$
$$\hat{\beta} := \beta + \sum_{i=1}^{N} 1 - x_i$$



- The **gamma** and **digamma** functions are used in computing statistics associated with the Beta distribution and are given below

$$\Gamma(n) = (n-1)! \qquad \psi(n) = \frac{\mathrm{d}\Gamma(n)/\mathrm{d}n}{\Gamma(n)} \qquad \text{for } n \in Z^+$$

Bernoulli Trader Model

- We assume traders are myopic, risk-neutral, and rational
- Trader **informativeness** is modeled by N, the size of their private sample of Bernoulli observations at each trade

$$x_1,\ldots,x_N \overset{iid}{\sim} \text{Bern}(p)$$

- Traders may also have limiting **budgets** B, modeled by an additional constraint of using the largest observation sub-sequence of length k where worst-case losses do not exceed B

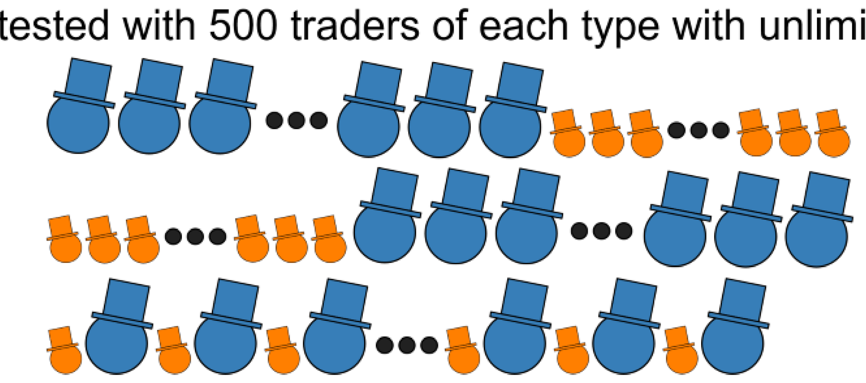
$$\max_{p} \left\{ C\left(\eta + \left[\sum_{i=1}^{k} x_i, k - \sum_{i=1}^{k} x_i\right]^T\right) - C(\eta) - \left[\sum_{i=1}^{k} x_i, k - \sum_{i=1}^{k} x_i\right] \cdot [\ln p, \ln(1-p)] \right\} \leq B$$

- Trades proceed as follows:
  - Trader **samples from Bernoulli** distribution according to their informativeness
  - Trader **updates Beta posterior beliefs** according to $\hat{\alpha} := \alpha + \sum_{i=1}^{N} x_i$, $\hat{\beta} := \beta + \sum_{i=1}^{N} 1 - x_i$
  - Trader purchases shares such that **market posterior matches private beliefs**

Informativeness and Sequence Results

- **Experimental Setup** High Informativeness ↔ $N = 5$ Low Informativeness ↔ $N = 1$

Three sequences are tested with 500 traders of each type with unlimited budgets



- We report average compensation for each trader type (Table 1) and overall belief aggregation (Fig. 1) over each of the three above sequences and random ordering

p_{true}	Ordering	$10^3\times$ (Avg HI)	$10^3\times$ (Avg LI)
0.25	HI First	6.65 ± 0.03	0.19 ± 0.02
	LI First	1.77 ± 0.04	5.05 ± 0.03
	Interleaved	5.55 ± 0.05	1.31 ± 0.04
	Random	5.60 ± 0.05	1.22 ± 0.04
0.50	HI First	6.39 ± 0.03	0.17 ± 0.02
	LI First	1.77 ± 0.04	4.79 ± 0.03
	Interleaved	5.49 ± 0.04	1.04 ± 0.04
	Random	5.48 ± 0.04	1.07 ± 0.04

Table 1: Net compensations of HI and LI traders, averaged over all traders of each type and over 10^4 simulations

LI traders receive higher compensation on average than HI traders when LI traders arrive first

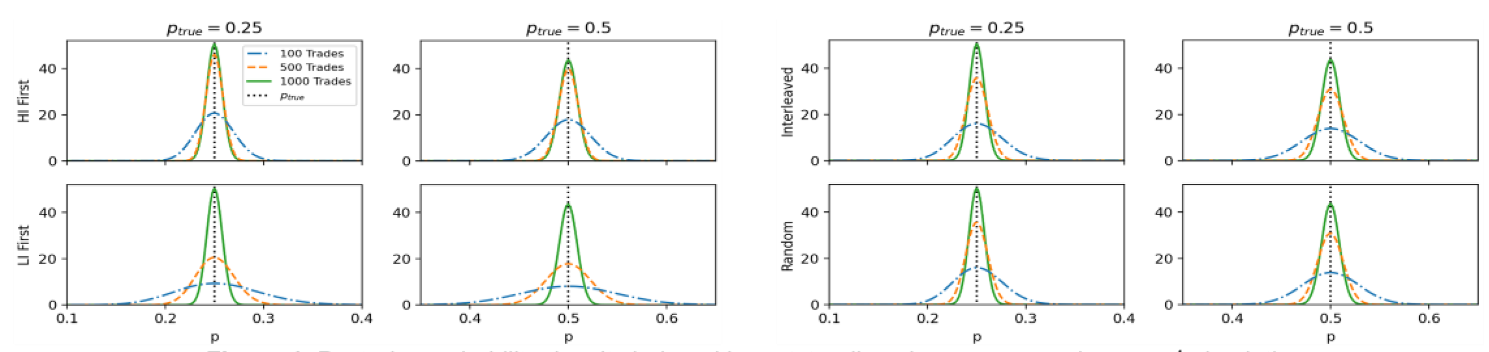


Figure 1: Posterior probability density induced by outstanding shares averaged over 10^4 simulations

Posterior distribution (on average) converges to p_{true} in as early as 100 trades regardless of trader ordering

Beta Market Mechanism

- **Securities and payoffs:** The market has two securities with payoffs given by $\ln p_{true}$ and $\ln(1-p_{true})$ when $p_{true} \in [0,1]$, the true Beta forecast variable corresponding to the Bernoulli success parameter, is revealed
- **Outstanding shares and aggregated belief:** The market maintains and displays outstanding shares $\eta = [\eta_1, \eta_2]$ which also corresponds to market belief given by $\text{Beta}(p;\alpha,\beta)$ where $\eta_1 = \alpha - 1, \eta_2 = \beta - 1$. This gives us expectations of the Bernoulli parameter and payoffs

$$\mathbb{E}_\eta[p] = \frac{\eta_1+1}{\eta_1+\eta_2+2} \qquad \mathbb{E}_\eta[\ln p] = \psi(\alpha) - \psi(\alpha+\beta)$$
$$\mathbb{E}_\eta[\ln(1-p)] = \psi(\beta) - \psi(\alpha+\beta)$$

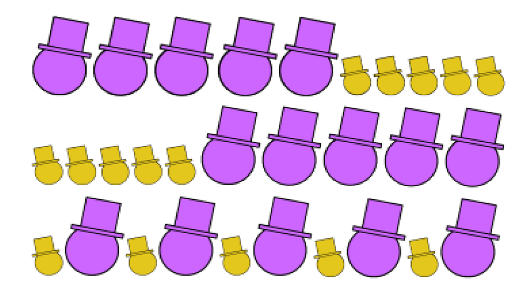
- **Cost Function:** This gives us the cost of a trade $C(\eta_{new}) - C(\eta_{old})$, the prices of each security $\nabla_\eta C(\eta)$, and is given by

$$C(\eta) = \ln\Gamma(\eta_1+1) + \ln\Gamma(\eta_2+1) - \ln\Gamma(\eta_1+\eta_2+2)$$

Budget and Sequence Results

- **Experimental Setup** Unlimited Budget ↔ $B_0 = \infty$ Limited Budget ↔ $B_0 \in (0,\infty)$

Three sequences are tested with 5 traders of each type with constant informativeness ($N = 5$ per trade) and initial budgets B_0. There are **25 successive market instances** where agents trade, receive compensation, and update their budgets before proceeding to the next round.



- We report the change in average LB trader budget over market rounds (Fig. 2) and impact of internal ordering among the LB traders (Fig. 3) over each of the three above sequences and random ordering

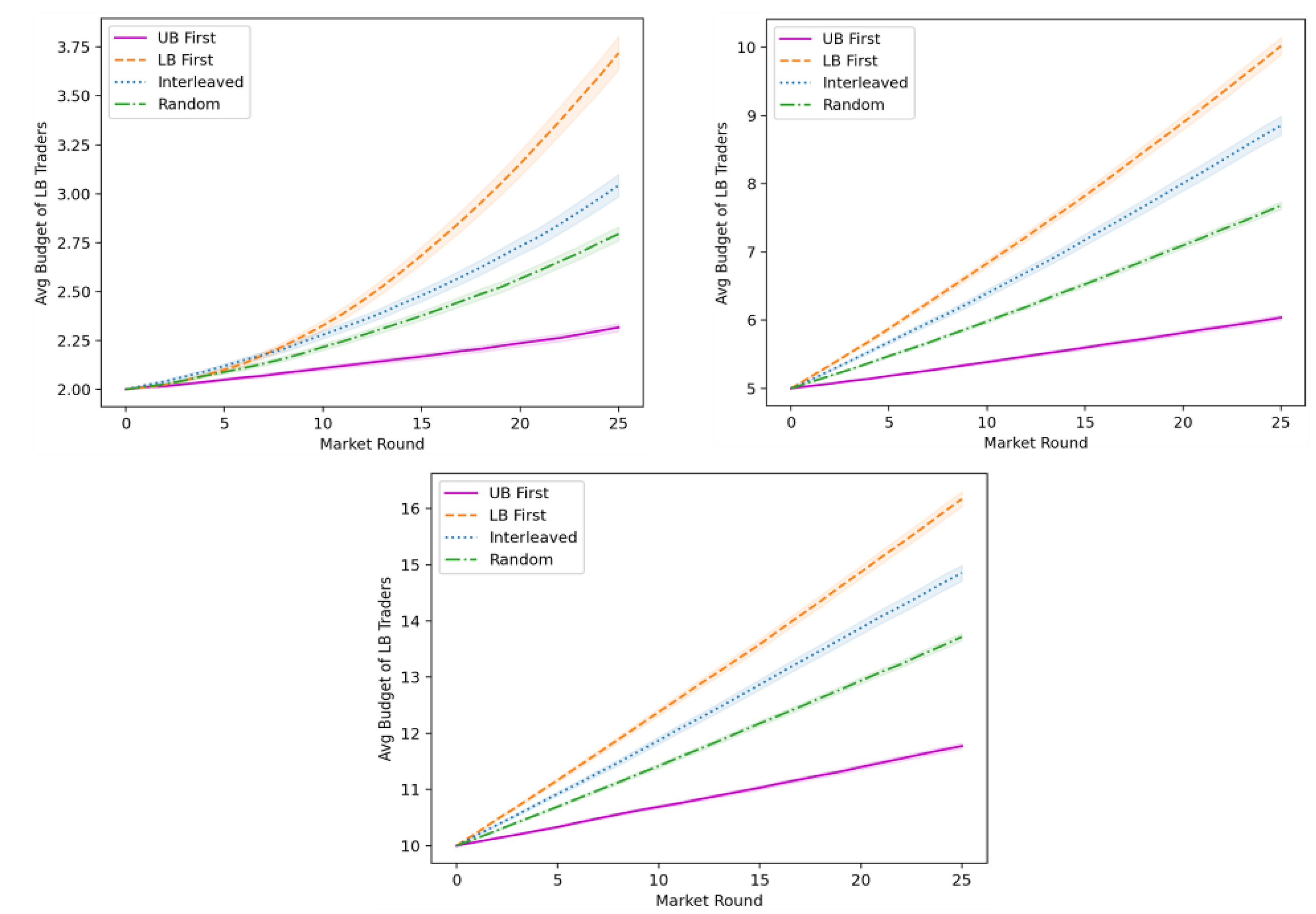


Figure 2: Change in budget of LB traders with an initial budget of $B_0 = 2$ (top left), $B_0 = 5$ (top right), and $B_0 = 10$ (bottom) over rounds 1 – 25, averaged over 10^4 simulations with $p_{true} \sim U[0.05, 0.95]$ for different trader orderings.

Growth in LB trader budget appears convex with non-monotonic dependence on initial budget

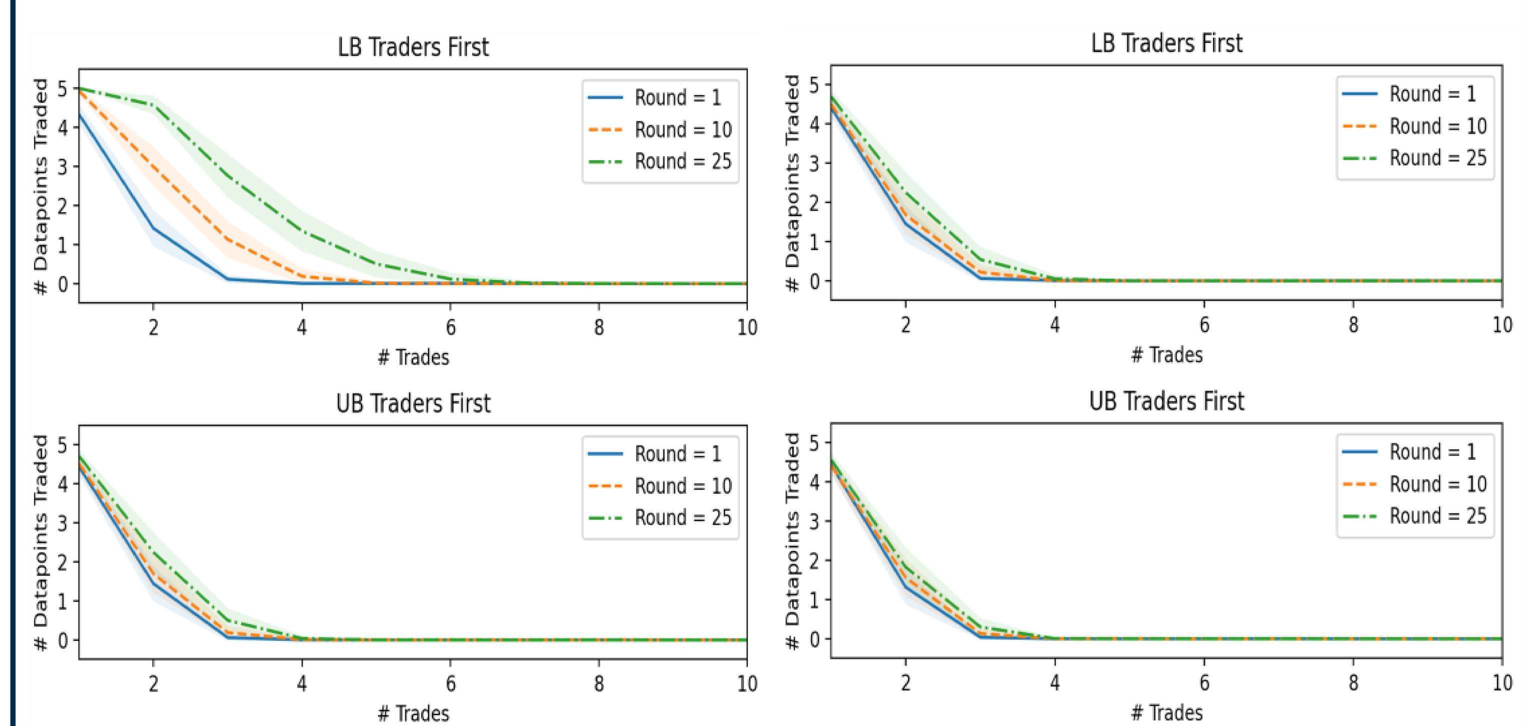


Figure 3: Average number of datapoints used to make a budget-constrained trading decision by the first (left) and last (right) trader in the subsequence of multi-shot LB traders over 10^3 simulations with $B_0 = 10$

All LB traders can trade on more information in later rounds, though this increase is significantly larger for the earliest LB traders